Financial Economics

6 Static Finance Economy

LEC, SJTU

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Overview

Arrow-Debreu
general equilibrium,
welfare theorem,
representative agent

Radner economies
real/nominal assets,
market span,
risk-neutral prob.,
representative good

von Neumann Morgenstern
measures of risk
aversion, HARA class

Finance economy
SDF, CCAPM,
term structure

Data and the Puzzles

Empirical resolutions

Theoretical resolutions

A Finance Economy

- Combines **Arrow-Debreu-Radner economy** with **vNM expected utility theory**
- Finance economy - built on GE theory + vNM utility theory
- Why do this?
  - To obtain empirically testable asset pricing formula
  - To study how much society is willing to pay for a marginal reduction of risk or the welfare implications of risks in the economy
- How to do this?
  - To apply vNM theory to a GE model we need
    - To treat assets as lotteries
    - Make assumptions that allow consideration of the utility of consumption now and tomorrow (two-period world)

Combining vNM EU theory and GE-I

- In vNM world, a risky decision is characterized by the agent' s vNM utility function v, her initial wealth (w), and the lottery $[x_1\pi_1, \ldots, x_S\pi_S]$ under review
- Objective function is:

$$\sum_{s=1}^{S} \pi_s v(w + x_s)$$

- If wealth is viewed as state contingent the objective is the expected utility of state-contingent wealth $(w + x_1, w + x_2, \ldots, w + x_S)$
- Adapting this with a GE model requires two things:
  - Capture uncertainty in the economy not by an arbitrary set of lotteries or gambles but
  - Instead used the idea of states of the world to model uncertainty.

Combining vNM EU theory and GE-II

- We restrict the set of vNM lotteries to include only lotteries with S possible outcomes and a fixed probability distribution over these outcomes
- Distribution over outcomes corresponds to the probability distribution of the states of the world
- Asset $\equiv$ a lottery that assigns different payoffs x_s to different states of the world
- Portfolio of assets $\equiv$ mixture of lotteries
- Our GE model specifies two periods:
  - Agents can choose how much to allocate to different states tomorrow (across states) but also how much to consume now and tomorrow (over time).

EU over two periods-I

- We do this by using a vNM utility function that is additively separable over time
- There is a vNM utility function:
  - v maps today' s consumption to today' s utility
  - u maps tomorrow' s consumption to tomorrow' s utility
- Total expected utility is:

$$v(y^0) + E[u(y)]$$

- Need to model how our utility function changes through time
- We assume that v and u are equal in terms of risk aversion- u is a linear transformation of v

EU over two periods-II

- We let: $u(y) = \delta v(y)$
- Here δ is the time preference parameter
- we also assume $\delta \leq 1$ or consumption today is valued more than consumption tomorrow
- Agent maximizes:

$$v(y^0) + \delta E[v(y)]$$

- So now we have:
  - An agent i with vNM utility v_i, impatience δ_i and state-contingent income tomorrow $w(i)$
  - If we let different agents in the economy have different beliefs $\pi(i)$
  - We also assume asset markets are complete i.e. the return matrix is invertible or it could be replaced by an identity matrix- there is a market for each Arrow security
  - Economy described by: $v_i, \delta_i, \pi(i), w(i); r$ the return matrix or now e: the Arrow security matrix

Portfolio problem- with vNM agents I

- Income tomorrow is uncertain or state-contingent (a lottery)
- Problem for the individual:
  - How much to save today?
  - How to insure your income tomorrow?
- Wish to move wealth through time (save or borrow)
- Wish to move across states (insure or take bets)
- Decision problem of a vNM agent in a finance economy is:

$$\max \left\{ v_i(y^0) + \delta_i E^i\{v_i(y)\} \;\middle|\; \begin{array}{ll} y^0 - w^0 \leqslant -q \cdot \tilde{z} & \\ y^s - w^s \leqslant r_s \cdot \tilde{z}^s & \text{for } s = 1, \ldots, S \end{array} \right\}$$

The Portfolio problem-II

- We can write this more compactly as:

$$\max\left\{v_i(y^0)+\delta_i E^i[v_i(y)]\;\middle|\;(y^0-w^0)+\sum_{s=1}^{S}\alpha_s(y^s-w^s)\leq 0\right\}$$

- For Equilibrium: W and Y is the aggregate endowment and consumption, respectively

$$W^s := \sum_{i=1}^{I} w^s(i), Y^s := \sum_{i=1}^{I} y^s(i), s = 0, 1, \ldots, S$$

- The Radner equilibrium of this asset economy is a pair consisting of a price for each asset α and an allocation $(y(1), \ldots, y(I))$ such that $y(i)$ solves the above optimization problem for each i and all markets clear or $Y - W = 0$

How do we deal with agent's beliefs?

- We have allowed for agents in our model to have different beliefs about the states of the world tomorrow
- Suppose there is a true objective probability distribution over the states of the world –but this is not known
- Each agent receives an imperfect signal about the true distribution
- Agent's signal is correlated with true distribution plus some noise
- Its thus important to know not only your distribution but also that of others in order to improve your ability to judge the true distribution
- Valuable to have your opinion- but also important to know other people's opinion

Common Beliefs-I

- Suppose every agent uses her own assessment of probabilities to maximize her expected utility - resulting equilibrium prices will contain information about the average opinion of the other agents
- Implies that every agent will want to revise their probability assessments
- We have allowed for agents in our model to have different beliefs about the states of the world tomorrow
- Our definition of equilibrium is incomplete - we need a combination of allocation prices and beliefs so that all markets clear and there is no incentive to revise beliefs
- How do we tackle this? - by simply assuming that everyone has the same beliefs

Common Beliefs-II

- In this equilibrium prices will be compatible with common beliefs - no one will need to revise them
- Fully revealing rational expectations equilibrium (REE) - Radner equilibrium of an economy with heterogeneous beliefs - which is such that market prices are a sufficient statistic for all information of all agents
- This is a strong assumption but not making it lead to much more complicated models because prices have two roles:
  - Measuring scarcity of goods
  - Conveying private information to the public
- We assume that "market prices" are a sufficient statistic for the information of all agents
- Agents simply use this common information and ignore private information completely
- This is called a Fully Revealing REE in the literature

Common Beliefs-III

- In this equilibrium it seems rational for an agent to use the commonly available information in decision-making and to completely ignore his private information
- Suppose all agents ignored private information - then how can prices be affected i.e. by information that everyone has - this is the Grossman-Stiglitz paradox - we will not pursue this here
- We now assume common beliefs and define beliefs π as a part of the economy not as a property of an agent
- From now on an agent with intertemporal vNM utility is a triple $(v_i, \delta_i, w(i))$ and an economy is the collection of all agents plus beliefs π and an asset matrix r

Risk Sharing

- Suppose two states have the same aggregate endowment - though they may differ with respect to the state-contingent distribution of income among agents
- Such states differ only with respect to **idiosyncratic risk** - no aggregate risk between them - an efficient allocation implies that everyone should consume the same in both states
- Mutuality Principle (Wilson 1968)
  - An efficient allocation of resources requires that only aggregate risk be borne by the agents - all idiosyncratic risk can be diversified away by mutual insurance among agents
  - Agents should only bet on aggregate risk - an individual's consumption is a function of aggregate endowment only

Risk Sharing in Edgeworth Box

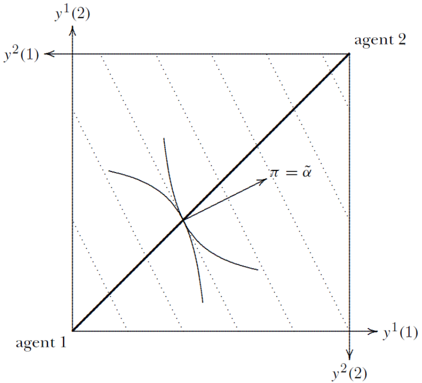

Figure 5.1. *An Edgeworth box with no aggregate risk and full insurance. The dotted lines are iso-expected wealth lines, the fat line is the contract curve.*

- The equilibrium prices are just collinear to the probabilities, $\alpha_s = \lambda \pi_s$
- In equilibrium, if there is no aggregate risk, then risk-neutral probabilities $\tilde{\alpha} = \pi$

Allocation of Aggregate Risk among Agents

- We know from the mutuality principle that in an efficient allocation people bear only aggregate risk
- But who bears this risk - How is the burden of aggregate risk allocated among the agents?
- We can get an insight by looking at the weights in the Social Welfare function
- We know from the SWF that for every Pareto efficient allocation there is a vector of weights one for each agent - with vNM agents and common beliefs the SWF is:

$$V(z) \coloneqq \max \left\{ \frac{1}{I} \sum_i \sigma_i \left[v_i(y^0(i)) + \delta_i E[v_i(y(i))] \right] \middle| \sum_i (y(i) - z) \leq 0 \right\}$$

The Social Welfare Function

- This objective function is additively separable between states and there is one constraint for each state
- We can thus write this problem as a sum of simple one-dimensional maximization problems

$$V(z) := \underbrace{\max\left\{\frac{1}{I}\sum_i \sigma_i v_i(y^0(i)) \,\middle|\, \sum_i (y^0(i) - z^0) \leqslant 0\right\}}_{=:v(z^0)}$$

$$+ \sum_{s=1}^{S} \pi_s \underbrace{\max\left\{\frac{1}{I}\sum_i \sigma_i \delta_i v_i(y^s(i)) \,\middle|\, \sum_i (y^s(i) - z^s) \leqslant 0\right\}}_{=:u(z^s)}$$

$$= v(z^0) + E\{u(z)\}.$$

Risk Sharing in Edgeworth Box

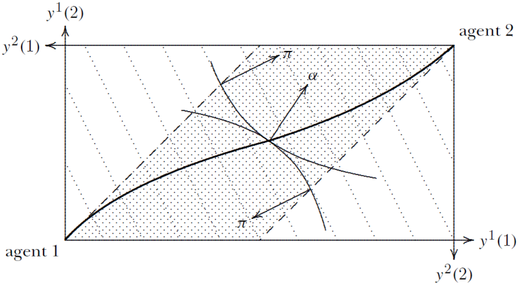

Figure 5.2. *The biotope of the contract curve.*

- The convex shape of the indifference curves implies that if they are tangent somewhere it will be in the shaded area
- This means that both agents bear some of the aggregate risk

Second Result of Wilson (1968)

- Consider

$$u(z) := \max_{y(i)} \left\{ \frac{1}{I} \sum_i \sigma_i \delta_i v_i(y(i)) \middle| \sum_i (y(i) - z) \leq 0 \right\}$$

- The FOC is:

$$\frac{1}{I} \sigma_i \delta_i v_i'(y(i)) = \mu$$

- Here μ is the LM of the feasibility constraint - measures the marginal increase in u when the constraint is marginally eased - expanding z by dz and eases the constraint I times dz; thus $u'(z) = I\mu$ and therefore $\sigma_i \delta_i v_i'(y(i)) = u'(z)$
- Totally differentiating yields $\sigma_i \delta_i v_i''(y(i)) dy(i) = u''(z) dz$

Wilson's Theorem-II

- Solving for i's marginal share of the aggregate risk, $dy(i)/dz$, yields:

$$\frac{dy(i)}{dz} = \frac{u''(z)}{\sigma_i \delta_i v_i''(y(i))}$$

- But $\sigma_i \delta_i = u'(z)/v_i''(y(i))$; thus

$$\frac{dy(i)}{dz} = \frac{u''(z)}{u'(z)} \cdot \frac{v_i'(y(i))}{v_i''(y(i))} = \frac{T_i(y(i))}{T(z)}$$

- Where T_i is i's absolute tolerance and T is the tolerance associated with the utility function u - the marginal share of the aggregate risk borne by an agent i is proportional to the agent's absolute risk tolerance

Wilson's Theorem-II

- Feasibility requires that the average change of consumption $dy(i)$ equals the change of per capita endowment dz
- Taking averages of the following we get:

$$\frac{dy(i)}{dz} = \frac{T_i(y(i))}{T(z)}$$

$$T(z) = \frac{1}{I}\sum_{i=1}^{I} T_i(y(i))$$

- The risk tolerance of u is the average risk tolerance of the population.
- Wilson (1968) result: The marginal aggregate risk borne by an agent equals the ratio of his absolute risk tolerance to the average risk tolerance of the population

A Risk-neutral Representative NM Agent

- Consider a one-person economy $((v, \beta, W/I), \tilde{\alpha}, r)$
  - Risk-neutral NM utility function: $v(y) := y$
  - The time-preference is given by the price of a risk-free bond $\beta := \sum_{s=1}^{S} \alpha_s$
  - The beliefs are the risk-neutral probabilities $\tilde{\alpha}_s := \alpha_s / \beta$
  - W denotes the aggregate state-contingent income
- The maximization problem of this single agent is

$$\max \left\{ y^0 + \beta \sum_{s=1}^{S} \tilde{\alpha}_s y^s \,\middle|\, (y^0 - W^0/I) + \sum_{s=1}^{S} \alpha_s (y^s - W^s/I) \leq 0 \right\}$$

- $(\alpha, W/I)$ is an equilibrium of this economy
- It becomes clear why $\tilde{\alpha}$ are called risk-neutral probabilities: they are the beliefs of the risk-neutral representative

Social Risk Preference

- We can also generate a local representative via the intertemporal NM social welfare function

$$v(z) := \max \left\{ \frac{1}{I} \sum_i \lambda_i^{-1} v_i(y(i)) \middle| \sum_i (y(i) - z) \leq 0 \right\}$$

$$u(z) := \max \left\{ \frac{1}{I} \sum_i \lambda_i^{-1} \delta_i v_i(y(i)) \middle| \sum_i (y(i) - z) \leq 0 \right\}$$

- A NM agent with utility v today and utility u tomorrow and mean per capita endowment is a NM representative
- By Wilson' s Theorem, we know that the absolute risk tolerance of this representative, for risk borne tomorrow (i.e. the risk tolerance of utility u), is equal to the mean absolute risk tolerance of the population as a whole

Social Time Preference

- Can we say something similar about society's time preference? Can we compute a δ such that $u(z) = \delta v(z)$?
- Such a δ would have to satisfy

$$\delta = \frac{u(z)}{v(z)} = \frac{\max\left\{ \frac{1}{I}\sum_i \lambda_i^{-1}\delta_i v_i(y(i)) \,\middle|\, \sum_i (y(i) - z) \leq 0 \right\}}{\max\left\{ \frac{1}{I}\sum_i \lambda_i^{-1} v_i(y(i)) \,\middle|\, \sum_i (y(i) - z) \leq 0 \right\}}$$

- In general social time preference is not well defined
- In the special case where everyone in the population has the same time preference, $\delta_1 = \cdots = \delta_I$, the representative has this same common time preference δ

Distribution Independent Aggregation

- The RA's tastes depend on all aspects of the economy including inter-personal income distribution
- Thus asset prices will depend not only on aggregate endowment but on the distribution as well
- What assumptions are required to make the representative agent's utility independent of the distribution?
- One case in which this is possible is if there is no aggregate risk
  - There is a risk-neutral representative agent whose beliefs are equal to the objective probabilities
  - Likewise, suppose there is some aggregate risk, but there is also a group of risk-neutral agents who are jointly rich enough to be able to absorb the whole aggregate risk
  - This follows from Wilson's Theorem as well: the risk-neutral agents are infinitely risk tolerant, $T_i(y(i)) = +\infty$; thus, average (= representative) risk tolerance is also infinite, and the representative is risk-neutral, no matter what the income distribution is

Distribution Independent Aggregation

- Rubinstein (1974) shows that if individuals have HARA utility with a quantity defined as common cautiousness then the RA's utility does not depend on the distribution of income
- Suppose that everyone has HARA utility, possibly with different constants a_i and cautiousness parameters b_i
- Then, by Wilson' s theorem, risk tolerance of the representative is given by

$$T(W^S/I) = \frac{1}{I}\left[\sum_i a_i + \sum_i b_i y^s(i)\right]$$

- If all agents have the same cautiousness $b_1 = \cdots = b_I$, however, then the representative's cautiousness will equal this common individual cautiousness
- In that case, the representative' s utility no longer depends on the distribution and his cautiousness is independent of the state; i.e., the representative is HARA

Distribution Independent Aggregation

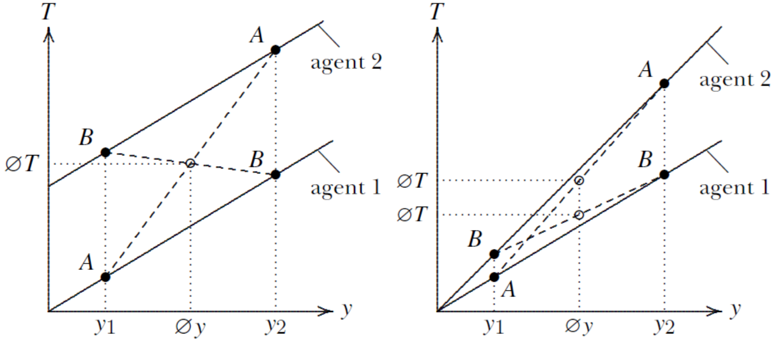

Figure 5.3. *Why the common cautiousness assumption is needed.*

Stochastic Discount Factor

- The stochastic discount factor, or SDF, is defined as

$$M_s := \frac{\alpha_s}{\pi_s}$$

- The SDF is positive if and only if there are no arbitrage opportunities; The SDF associated with an equilibrium is unique if and only if markets are complete

$$q_j = E[Mr^j]$$

$$E[MR^j] = 1$$

The SDF and the MRS

- The representative agent's portfolio problem is:

$$\max\left\{v(y^0)+\delta E[v(y)]\,\middle|\,(y^0-w^0)+\sum_{s=1}^{S}\alpha_s(y^s-w^s)\leq 0\right\}$$

- We know that the equilibrium net trade of the representative is zero. Thus the FOC must be satisfied at the endowment point:

$$\delta\pi_s\frac{v'(w^s)}{v'(w^0)}=\alpha_s$$

- Thus if (v,δ,w) is a vNM representative then in equilibrium:

$$M_s:=\frac{\alpha_s}{\pi_s}=\delta\frac{v'(w^s)}{v'(w^0)}$$

- Stochastic discount factor (SDF) is the MRS $\times$ rate of time preference - stochastic because it is so + discount because it is like a PV factor

Risk-neutral v.s. Objective Probabilities

- We have $\alpha_s = \pi_s M_s$ and $\tilde{\alpha}_s = \rho\alpha_s$, thus

$$\frac{\tilde{\alpha}_s}{\pi_s} = \rho M_s = \frac{\delta}{\beta} \cdot \frac{v'(w^s)}{v'(w^0)}$$

  and we have

$$\beta = E[M] = \delta \frac{E[v'(w)]}{v'(w^0)}$$

  hence

$$\frac{\tilde{\alpha}_s}{\pi_s} = \frac{v'(w^s)}{E[v'(w)]}$$

- Then the risk-neutral probability distribution is pessimistic in the sense that it puts excessive weight on low-income states, and little weight on high-income states

The equilibrium price of time

- Consider first an economy with no uncertainty and no growth, so that income tomorrow is the same as income today and is independent of the state of the world. Then

$$\beta = \delta \frac{v'(w^0)}{v'(w^0)} = \delta$$

- Suppose now there is growth, but still no uncertainty, so $w^s := (1+g)w^0$, for $s = 1, \ldots, S$. $g > 0$ is the growth rate of income. Then

$$\beta = \delta \frac{v'((1+g)w^0)}{v'(w^0)} < \delta$$

- Hence, with growth, the price of a risk-free bond is smaller than without growth, or, equivalently, the risk-free interest rate is greater with growth

The equilibrium price of time

- Suppose now that again there is no growth, and add uncertainty in the form of a mean-preserving spread, i.e. $\exists(s,s')w^s \neq w^{s'}$,but $E[w] = w^0$
- Suppose that v' is a linear function. In that case, the mean-preserving spread of income has no effect on β
- Suppose the representative agent is prudent ($v''' > 0$, and v' is a convex function), then the corresponding risk-free interest rate decreases

Manipulating the SDF-Equilibrium Price of Risk

- Now we show that simple manipulation of the fundamental asset pricing equation gives us a range of insights
- Start with the covariance decomposition:

$$1 = E[MR^j] = E[M]E[R^j] + cov(M, R^j) = \beta E[R^j] + cov(M, R^j)$$

- Consumption-based capital asset pricing model, CCAPM: In equilibrium the SDF is given by the FOC of the portfolio problem of the representative:

$$E[R^j] - \rho = \rho cov(-M, R^j) = \frac{cov(-v'(w), R^j)}{E[v'(w)]}$$

Equilibrium Price of Risk-I

- If the rate of return of an asset is not correlated with aggregate risk then the risk premium is zero and the expected return on this asset equals the risk-free rate. Why?
- Any risk inherent in this asset can be diversified away since it is not related to aggregate risk.
- The risk of this asset will not be borne by anyone in an efficient allocation (by the Mutuality Principle) and thus has no effect on the price of the asset.
- An asset whose return covaries positively with aggregate endowment will carry a positive risk premium. Why?
- This asset pays out in good times and fails to pay off in bad times - thus we need an incentive or a positive risk premium to hold such an asset.

Equilibrium Price of Risk-II

- An asset whose return covaries negatively with aggregate endowment is a hedge against aggregate risk - it can be used to ensure against aggregate risk.
- Of course such insurance is not possible for the aggregate but this asset allows the owner to pass on the aggregate risk to other agents.
- Hence such assets are valuable and carry a negative risk premium.

Special Cases: No aggregate risk or risk-neutral representative agent

- If the representative agent's income is constant in all states, $w^1 = \cdots = w^S$, the stochastic discount factor is a constant and equals $M_s = \delta v'(w^1)/v'(w^0) = \beta$
- The price of an asset with returns rj is therefore simply

$$q_j = \beta E[r^j]$$

- Similarly, using the CCAPM, all assets have the same expected return rate in that case:

$$E[R^j] = \rho$$

- If the representative agent is risk neutral, the stochastic discount factor is degenerate and equals the plain discount factor δ in all states

Special Cases: Quadratic utility representative agent and the CAPM

- Suppose there is a special asset, $R^m_s = -av'(w^s) + b$, $a > 0$, then the CCAPM formula can be written as

$$E[R^j] - \rho = \frac{cov(R^m, R^j)/a}{E[v'(w)]}$$

- Evaluated for $j = m$ can help us get rid of the v'

$$\frac{E[R^j] - \rho}{E[R^m] - \rho} = \frac{cov(R^m, R^j)}{var(R^m)}$$

- Defining $\beta_j := cov(R^m, R^j)/var(R^m)$ yields

$$E[R^j] = \rho + \beta_j[E[R^m] - \rho]$$

- This equation is known as the capital asset pricing model, or CAPM (Sharpe, 1964)

Special Cases: Quadratic utility representative agent and the CAPM

- Let m be a claim on aggregate or mean endowment, so that $r_s^m = w^s$. Let q_m be the price of this asset; then $R_s^m = w^s / q_m$
- Suppose further that the utility function of the representative agent is quadratic, i.e. $v(y) := -cy^2 + dy$
- In this case R^m is perfectly negatively correlated with marginal utility, $R_s^m = -av'(w^s) + b$, with $a := [2cq_m]^{-1}$ and $b := ad$
- Hence, with quadratic utility, the CCAPM collapses to the CAPM
- The CAPM is a special case of the CCAPM

Special Cases: CRRA representative

- Suppose now the representative agent has constant relative risk aversion $v(y) = y^{1-\gamma}/(1-\gamma)$
- Defining the state-contingent growth rate of per capita income as $1 + g_s := w^s/w^0$, then $M_s = \delta(1+g_s)^{-\gamma}$, or in logs,

$$\ln M_s = \ln \delta - \gamma \ln(1+g_s) \approx \ln \delta - \gamma g_s$$

- Since $\beta = \rho^{-1} = E[M]$, we have

$$\ln \rho = -\ln E[M] \approx E[\ln M] \approx \gamma E[g] - \ln \delta$$

- The risk-free interest rate is an affine function of the expected growth rate

Special Cases: CRRA representative

- Consider the equilibrium risk premium. Substituting the CRRA utility into the CCAPM formula yields

$$E[R^j] - \rho = \rho\delta cov(-(1+g)^{-\gamma}, R^j)$$

- When g is relatively small, we can approximate $(1+g)^{-\gamma}$ with $1 - \gamma g$; thus

$$E[R^j] - \rho \approx \rho\delta\gamma cov(g, R^j)$$

- Replace $[\rho\delta]^{-1} = E[v'(w)]/v'(w^0) = E[(1+g)^{-\gamma}] \approx 1 - \gamma E[g]$, we have

$$E[R^j] - \rho \approx \gamma^* cov(g, R^j), \text{ with } \gamma^* := \frac{\gamma}{1 - \gamma E[g]} \approx \gamma$$

Static Finance Economy

- Combining general equilibrium with vNM agents
- Two principles for efficient risk-sharing
  - Mutuality principle
  - Wilson's second theorem: marginal aggregfate risk borne proportional to absolute risk tolerance
- Aggregate representative vNM agents
- The stochastic discount factor
- Consumption-based capital asset pricing model, (CCAPM)