Financial Economics

5 Portfolio Choice and Asset Pricing

LEC, SJTU

2024 Winter

Overview

- Mean-variance Utility
- Mean-variance Analysis and Portfolio Choice
- Capital Asset Pricing Model

Mean-Variance Utility

- Many researchers in finance (Markowitz, Sharpe etc.) used mean-variance utility functions. But is it compatible with vNM theory?
- The answer is yes ... approximately ... under some conditions
- What are these conditions?
  - v is quadratic
  - If asset returns are joint normal
  - For small risks

Mean-Variance: Quadratic Utility

- Suppose utility is quadratic, $v(y) = ay - by^2$
- Expected utility is then

$$E[v(y)] = aE[y] - bE[y^2] = aE[y] - b(E[y]^2 + Var(y))$$

- Thus, expected utility is a function of the mean, $E[y]$, and the variance, $Var(y)$, only
- This function increases monotonically in the mean as long as $E[x] < a/2b$, and it decreases monotonically in the variance
- This justification for mean–variance analysis is not a good one, though, because quadratic utility implies IARA

Mean-Variance: Joint Normal

- Suppose all lotteries in the domain have normally distributed prized. (They need not be independent of each other)
  - This requires an infinite state space
- It is a fact of mathematics that any combination of such lotteries will also be normally distributed
- The normal distribution is completely described by its first two moments
- Therefore, the distribution of any combination of lotteries is also completely described by just the mean and the variance
- As a result, expected utility can be expressed as a function of just these two numbers as well

Mean-Variance: Small Risks

- The most relevant justification for mean-variance is probably the case of small risks
- If we consider only small risks, we may use a second order Taylor approximation of the vNM utility function
- A second order Taylor approximation of a concave function is a quadratic function with a negative coefficient on the quadratic term
    - In other words, any risk-averse NM utility function can locally be approximated with a quadratic function
    - But the expectation of a quadratic utility function can be evaluated with the mean and variance. Thus, to evaluate small risks, mean and variance are enough

Mean-Variance: Small Risks

- Let $f : R \rightarrow R$ be a smooth function. The Taylor approximation is

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!}$$

$$+f'''(x_0)\frac{(x - x_0)^3}{3!} + \cdots$$

- So $f(x)$ can approximately be evaluated by looking at the value of f at another point x_0, and making a correction involving the first n derivatives
- We will use this idea to evaluate $E[u(y)]$

Mean-Variance: Small Risks

- Consider first an additive risk, i.e. $y = w + x$ where x is a zero mean random variable
- For small variance of x, $E[v(y)]$ is close to $v(w)$
- Consider the second order Taylor approximation,

$$E[v(w+x)] \approx v(w)+v'(w)E[x]+v''(w)\frac{E[x^2]}{2} = v(w)+v''(w)\frac{Var(x)}{2}$$

- Let c be the certainty equivalent, $v(c) = E[v(w + x)]$
- For small variance of x, c is close to w, but let us look at the first order Taylor approximation

$$v(c) \approx v(w) + v'(w)(c - w)$$

Mean-Variance: Small Risks

- Since $E[v(w+x)] = v(c)$, this simplifies to

$$w - c \approx A(w)\frac{Var(x)}{2}$$

- w–c is the risk premium
- We see here that the risk premium is approximately a linear function of the variance of the additive risk, with the slope of the effect equal to half the coefficient of absolute risk

Mean-Variance: Small Risks

- The same exercise can be done with a multiplicative risk
- Let $y = gw$, where g is a positive random variable with unit mean
- Doing the same steps as before leads to

$$1 - \kappa \approx R(w)\frac{Var(g)}{2}$$

- where κ is the certainty equivalent growth rate, $v(\kappa w) = E[v(gw)]$
- The coefficient of relative risk aversion is relevant for multiplicative risk, absolute risk aversion for additive risk

Feasible Combination of Mean and Variance

- Consider an arbitrary portfolio $z = (z_1, \ldots, z_J)$. Associated with such a portfolio is a state-contingent wealth vector

$$W_s(z) = \sum_{j=1}^{J} r_s^j \cdot z_j, \quad s = 1, \ldots, S$$

- For a given probability vector $p = (p_1, \ldots, p_S)$ one can compute the mean pay-off $\mu(z)$ which is achieved by this portfolio z as

$$\mu(z) = \sum_{s=1}^{S} p_s \cdot W_s(z) = \sum_{s=1}^{S} p_s \cdot \left(\sum_{j=1}^{J} r_s^j \cdot z_j \right)$$

$$= \sum_{j=1}^{J} \sum_{s=1}^{S} p_s \cdot r_s^j \cdot z_j = \sum_{j=1}^{J} \mu_j \cdot z_j$$

- where $\mu_j := \sum_{s=1}^{S} p_s \cdot r_s^j$ denotes the expected pay-off of asset j

Feasible Combination of Mean and Variance

- Similarly, one can compute the variance of the pay-offs from portfolio z, $\sigma^2(z)$. Let $\sigma_{jk} := \sum_{s=1}^{S} p_s \cdot (r_s^j - \mu_j)(r_s^k - \mu_k)$ be the covariance of the pay-offs from assets j and k, then:

$$\sigma^2(z) = \sum_{s=1}^{S} p_s \cdot (W_s(z) - \mu(z))^2 = \sum_{s=1}^{S} p_s \left(\sum_{j=1}^{J} (r_s^j - \mu_j) \cdot z_j \right)^2$$

$$= \sum_{j=1}^{J} z_j \cdot \sum_{k=1}^{J} z_k \cdot \left[\sum_{s=1}^{S} p_s \cdot (r_s^j - \mu_j) \cdot (r_s^k - \mu_k) \right] = \sum_{j=1}^{J} \sum_{k=1}^{J} z_j \cdot z_k \cdot \sigma_{jk}$$

Feasible Combination of Mean and Variance

- The mean of a portfolio is the weighted sum of the mean pay-offs of the individual assets
- The variance of a portfolio is the quadratic form obtained from the matrix of covariances of asset pay-offs:

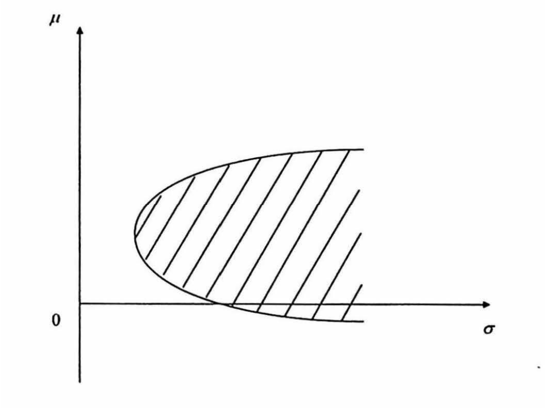
$$\sigma^2(z) = z \cdot \Omega \cdot z, \quad \text{where } \Omega := \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{J1} \\ \vdots & \ddots & \vdots \\ \sigma_{1J} & \cdots & \sigma_{JJ} \end{bmatrix}$$

- The feasible set of (μ, σ^2) combinations is written formally as follows:

$$\left\{ (\mu(z), \sigma^2(z)) | \sum_{j=1}^{J} q^j \cdot z_j = W_0 \right\}$$

Feasible Combination of Mean and Variance

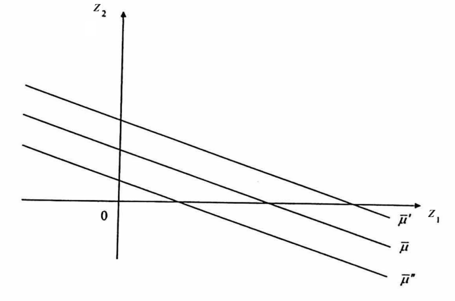
- It is possible to represent the set of feasible mean-variance combinations in a (μ, σ^2) diagram or, as is more common in the finance literature, in a mean-standard deviation diagram



Feasible Combination of Mean and Variance in a Two-Asset Model

- Suppose now we consider two assets, and it is possible to construct iso-μ and iso-σ contours in (z_1, z_2) space
- Iso-μ contours are linear with slope and location parameters μ_1 and μ_2:

$$\mu(z_1, z_2) := \mu_1 \cdot z_1 + \mu_2 \cdot z_2 = \bar{\mu}$$

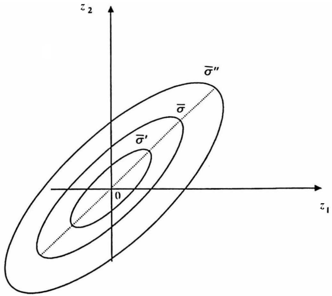


Feasible Combination of Mean and Variance in a Two-Asset Model

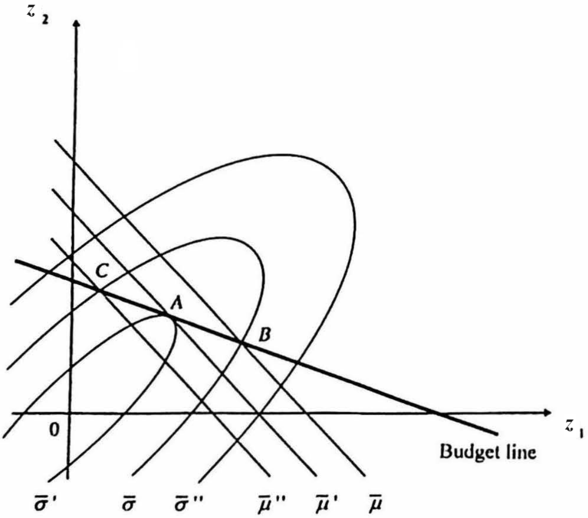
- Iso-σ contours are obtained by fixing a level of variance or standard deviation

$$\sigma^2(z_1, z_2) := \sigma_{11} \cdot z_1^2 + 2 \cdot \sigma_{12} \cdot z_1 \cdot z_2 + \sigma_{22} \cdot z_2^2 = \bar{\sigma}^2$$

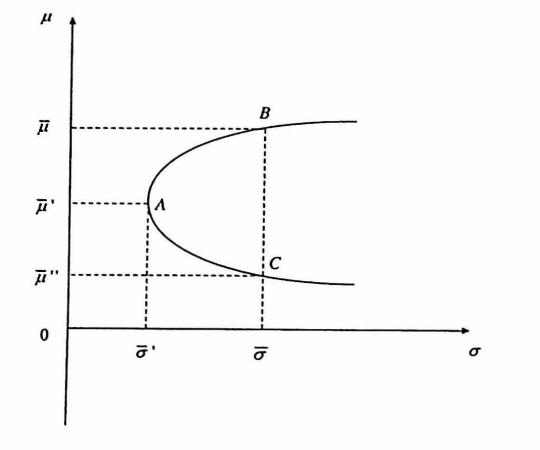
- Depending on the determinants of covariance matrix, there are two cases on the shape of Iso-σ contours:
  - if det $\Omega > 0$, the contour of $\sigma^2(z_1, z_2)$ must bean ellipse
  - if det $\Omega = 0$, the contour of $\sigma^2(z_1, z_2)$ must be a pair of parallel lines



Converting (z_1, z_2) Space to (μ, σ) Space



Converting (z_1, z_2) Space to (μ, σ) Space



- The budget line in (z_1, z_2) space has a unique representation in (μ, σ) space

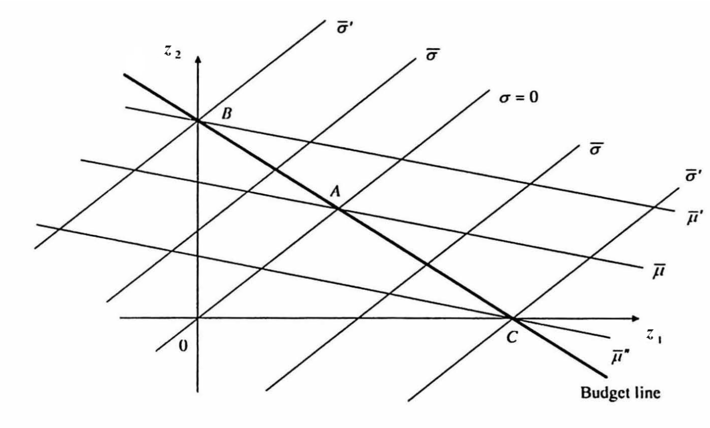
Minimum Variance Portfolio (MVP) in Complete Markets

- Two asset markets are complete if there are exactly two states of the world and the asset pay-offs are linearly independent
- Writing the probabilities of the two states as p and $(1-p)$, the standard deviation can be transformed to yield:

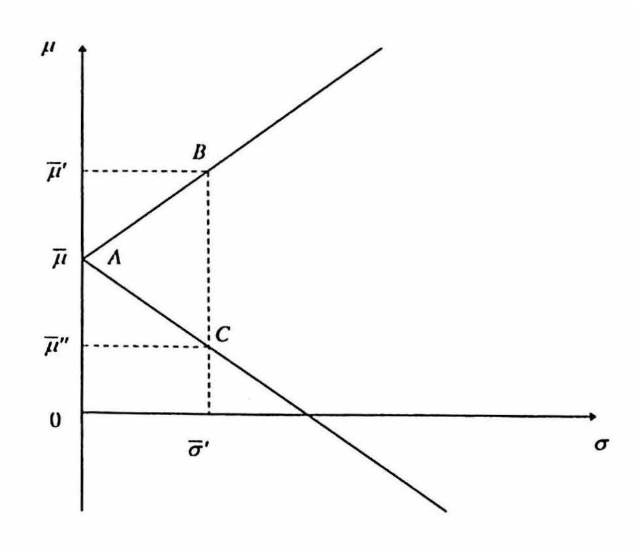
$$\sigma(z_1, z_2) = \sqrt{[p \cdot (1-p)]} \cdot |(r_1^1 - r_2^1) \cdot z_1 + (r_1^2 - r_2^2) \cdot z_2|$$

- The iso-σ contours are therefore also linear in (z_1, z_2) space
  - The iso-σ contours in the case of incomplete markets were ellipses centred on the origin and symmetric about a ray through the origin
  - In the case of complete markets, the ellipses are 'stretched out' infinitely in the direction of the longer of their two axes, and thus become a set of parallel straight lines

Minimum Variance Portfolio (MVP) in Complete Markets



Minimum Variance Portfolio (MVP) in Complete Markets



Minimum Variance Portfolio (MVP) in Complete Markets

Example: Let the pay-off matrix for the two assets be as follows:

$$\begin{bmatrix} r_1^1 & r_1^2 \\ r_2^1 & r_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 3/4 & 2 \end{bmatrix}$$

Let the probabilities of the two states be $(p, (1-p)) = (1/2, 1/2)$
Assume that the prices of the two assets are $(q_1, q_2) = (3/4, 1)$ and that initial wealth is $W_0 = 1$
Derive the mean-variance frontier and find the MVP

The iso-μ contours are given by the equation

$$z_2 = \frac{4}{5} \cdot \bar{\mu} - \frac{7}{10} \cdot z_1$$

The iso-σ contours are given by the equation

$$z_2 = \pm \frac{4}{3} \cdot \bar{\sigma} + \frac{1}{6} \cdot z_1$$

Minimum Variance Portfolio (MVP) in Complete Markets

Example: Let the pay-off matrix for the two assets be as follows:

$$\begin{bmatrix} r_1^1 & r_1^2 \\ r_2^1 & r_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 3/4 & 2 \end{bmatrix}$$

Let the probabilities of the two states be $(p, (1-p)) = (1/2, 1/2)$
Assume that the prices of the two assets are $(q_1, q_2) = (3/4, 1)$ and that initial wealth is $W_0 = 1$
Derive the mean-variance frontier and find the MVP

The equation for the budget line in (z_1, z_2) space is $(3/4) \cdot z_1 + z_2 = 1$
Solving the equations for the iso-μ and iso-σ contours to find a relationship between μ and σ, and substituting the relationship between z_1 and z_2 along the budget line, we obtain the mean-variance frontier:

$$\mu = \frac{13}{11} \pm \frac{1}{11} \cdot \sigma$$

And the MVP is found by substituting $\sigma = 0$ into the equation for the iso-σ contours: $(z_1, z_2) = (12/11, 2/11)$

Portfolio Choice in Mean-variance Space

- For the two-asset, two-state world, the agent's choice problem in (z_1, z_2) space is written formally as follows:

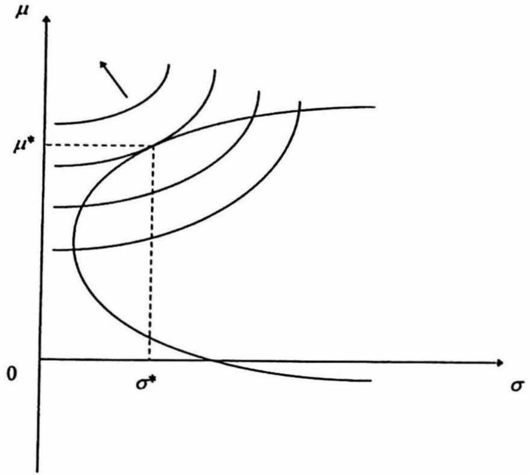
$$\max_{z_1, z_2} p \cdot u(r_1^1 \cdot z_1 + r_1^2 \cdot z_2) + (1-p) \cdot u(r_2^1 \cdot z_1 + r_2^2 \cdot z_2)$$

$$\text{subject to } q_1 \cdot z_1 + q_2 \cdot z_2 = W_0$$

- Alternatively, if there is a representation of the form $V(\mu, \sigma)$, then we can define the portfolio choice problem in mean-variance space

Portfolio Choice in Mean-variance Space

- The assumption that $V(\mu, \sigma)$ is increasing in μ and decreasing in σ implies positively sloped indifference curves



Portfolio Choice in Mean-variance Space

Example: Let's continue the previous example. Now we assume quadratic expected utility index be $u(W) := 4 \cdot W - (1/2) \cdot W^2$. Solve for the expected-utility-maximization problem in (z_1, z_2) space

The first order condition indicates that

$$\frac{p \cdot (4 - (r_1^1 \cdot z_1 + r_1^2 \cdot z_2)) \cdot r_1^1 + (1-p) \cdot (4 - (r_2^1 \cdot z_1 + r_2^2 \cdot z_2)) \cdot r_2^1}{p \cdot (4 - (r_1^1 \cdot z_1 + r_1^2 \cdot z_2)) \cdot r_1^2 + (1-p) \cdot (4 - (r_2^1 \cdot z_1 + r_2^2 \cdot z_2)) \cdot r_2^2} = \frac{q_1}{q_2}$$

Together with the budget constraint, we can derive that:

$$z_1 = 19 \cdot z_2 - 8, \quad z_2 = 1 - \frac{3}{4} \cdot z_1$$

Thus $(z_1^*, z_2^*) = (44/61, 28/61)$

Portfolio Choice in Mean-variance Space

Example: Let's continue the previous example. Now we assume quadratic expected utility index be $u(W) := 4 \cdot W - (1/2) \cdot W^2$. Consider the expected-utility-maximization problem in (μ, σ) space

The expected utility function can be transformed to yield:

$$V(\mu, \sigma) = 4 \cdot \mu - \frac{1}{2} \cdot \sigma^2 - \frac{1}{2} \cdot \mu^2$$

Therefore the slope of an indifference curve in (μ, σ) space is derived as:

$$\frac{d\mu}{d\sigma} = -\frac{\partial V(\cdot)/\partial \sigma}{\partial V(\cdot)/\partial \mu} = \frac{\sigma}{4 - \mu}$$

The slope of the (μ, σ) frontier, on the other hand, is given by the equation $\mu = \frac{13}{11} \pm \frac{1}{11} \cdot \sigma$
Thus we obtain $(\sigma^*, \mu^*) = (31/122, 147/122)$

The Capital Asset-Pricing Model

- Consider a general equilibrium in which there are I consumers, $(K-1)$ risky assets, and one risk-free asset
- Capital Asset-Pricing Model (CAPM) proves the surprising result that the relationship among the prices of assets in a general equilibrium is linear
- Definitions and notations:
  - The vector $z = (z_1, ..., z_K)$ represents a portfolio
  - The assets have pay-offs in each of the S states denoted $r_s^k (s = 1, \ldots, S; k = 1, \ldots, K)$
  - The wealth derived in each state: $W_s(z) = \sum_{k=1}^{K} r_s^k \cdot z_k$, with expectation $\mu(z) = \sum_{k=1}^{K} \mu_k \cdot z_k$ and variance: $\sigma^2(z) = \sum_{k=1}^{K} z_k \cdot \sum_{j=1}^{K} z_j \cdot \sigma_{jk}$
  - Denoting the partial derivative of $\mu(z)$ and $\sigma^2(z)$ w.r.t z_ℓ by $\mu_\ell(z) := \partial\mu(z)/\partial z_\ell$ and $\sigma_\ell^2(z) := \partial\sigma^2(z)/\partial z_\ell$
    $\mu_\ell(z) = \mu_\ell$ and $\sigma_\ell^2(z) = 2 \cdot [\sum_{k=1}^{K} z_k \cdot \sigma_{\ell k}] = 2 \cdot \sigma(z, \ell)$

The Capital Asset-Pricing Model

- Consider the optimization problem for some consumer $i \in 1, 2, \ldots, I$:

$$\max_z V^i(\mu(z), \sigma^2(z))$$

$$\text{subject to } \sum_{k=1}^{K} q_k \cdot z_k = \sum_{k=1}^{K} q_k \cdot \bar{z}_k$$

- The FOCs for this problem are:

$$V_1^i(\mu(z), \sigma^2(z)) \cdot \mu_\ell(z) + V_2^i(\mu(z), \sigma^2(z)) \cdot \sigma_\ell^2(z) = \lambda \cdot q_\ell, \text{ for } \ell = 1, \ldots, K$$

where $V_1^i(\cdot)$ and $V_2^i(\cdot)$ denote the partial derivatives of $V^i(\cdot)$ w.r.t μ and σ^2 and λ is the Lagrange multiplier

The Capital Asset-Pricing Model

- A general equilibrium in this exchange economy is a vector of asset prices $q^* = (q_1^*, \ldots, q_K^*)$ together with a vector of asset demands for each consumer $z^{i*} = (z_1^{i*}, \ldots, z_K^{i*})$ such that markets clear:

$$\sum_{i=1}^{I} z_k^{i*} = \sum_{i=1}^{I} \bar{z}_k^i := Z_k$$

- The capital asset-pricing equation is derived from the FOCs, evaluated at equilibrium, and assuming that one of the assets is riskless

The Capital Asset-Pricing Model

- Assuming that asset K is riskless, we know that $r_s^K = r$ for all $s = 1, \ldots, S$
- We have $\mu_K(z) = r$ and $\sigma_K^2(z) = \sigma(z, K) = 0$. Substituting these values into the first-order conditions and choosing the riskless asset as numeraire $q_K = 1$, we solve the K-th FOC as:

$$\lambda = V_1^i(\mu(z^{i*}), \sigma^2(z^{i*})) \cdot r$$

- Substituting for λ, the first $K - 1$ FOCs become:

$$V_1^i(\mu(z^{i*}), \sigma^2(z^{i*})) \cdot (\mu_\ell - q_\ell^* \cdot r) + 2 \cdot V_2^i(\mu(z^{i*}), \sigma^2(z^{i*})) \cdot \sum_{j=1}^{K} z_j^{i*} \cdot \sigma_{j\ell} = 0$$

The Capital Asset-Pricing Model

$$V_1^i(\mu(z^{i*}), \sigma^2(z^{i*})) \cdot (\mu_\ell - q_\ell^* \cdot r) + 2 \cdot V_2^i(\mu(z^{i*}), \sigma^2(z^{i*})) \cdot \sum_{j=1}^{K} z_j^{i*} \cdot \sigma_{j\ell} = 0$$

- This equation may be rewritten as:

$$\Theta^i(z^{i*}) \cdot (\mu_\ell - q_\ell^* \cdot r) = \sum_{j=1}^{K} z_j^{i*} \cdot \sigma_{j\ell}$$

  where $\Theta^i(z^{i*}) := -V_1^i(\mu(z^{i*}), \sigma^2(z^{i*}))/(2 \cdot V_2^i(\mu(z^{i*}), \sigma^2(z^{i*})))$ is the marginal rate of substitution along an individual agent's indifference curve in (μ, σ) space

The Capital Asset-Pricing Model

$$\Theta^i(z^{i*}) \cdot (\mu_\ell - q_\ell^* \cdot r) = \sum_{j=1}^{K} z_j^{i*} \cdot \sigma_{j\ell}$$

- Summing the equation over all consumers, and noting that $\sum_{i=1}^{I} z_k^{i*} = Z_k$ equilibrium (market clearing), we obtain:

$$\theta(z^*) \cdot (\mu_\ell - q_\ell^* \cdot r) = \sigma(Z, \ell)$$

where $\theta(z^*) := \sum_{i=1}^{I} \Theta^i(z^{i*})$ is the sum of the agents' marginal rates of substitution and $\sigma(Z, \ell) := \sum_{j=1}^{K} Z_j \cdot \sigma_{j\ell}$,is the covariance of asset ℓ with the aggregate endowments

The Capital Asset-Pricing Model

$$\theta(z^*) \cdot (\mu_\ell - q_\ell^* \cdot r) = \sigma(Z, \ell)$$

- Finally, multiplying the equation by Z_ℓ, and summing again over all risky assets $\ell = 1, \ldots, K-1$, we obtain

$$\theta(z^*) \cdot (\mu(Z) - r \cdot W_0(Z)) = \sigma^2(Z)$$

- We solve the equation for $\theta(z^*)$ and substituting into each assets' pricing equation:

$$\mu_\ell - r \cdot q_\ell^* = \frac{\sigma(Z, \ell)}{\sigma^2(Z)} \cdot [\mu(Z) - r \cdot W_0(Z)]$$

The Capital Asset-Pricing Model

- If we measure asset returns as pay-offs per unit invested and asset quantities in units of expenditure, we obtain the CAPM pricing formula usually used in finance literature

$$\hat{\mu}_\ell - r = \frac{\hat{\sigma}(Z, \ell)}{\hat{\sigma}^2(Z)} \cdot [\hat{\mu}(Z) - r]$$

- Writing $\beta_\ell := \hat{\sigma}(Z, \ell)/\hat{\sigma}^2(Z)$, pricing formula becomes:

$$\hat{\mu}_\ell - r = \beta_\ell \cdot [\hat{\mu}(Z) - r]$$

The Capital Asset-Pricing Model

- Purchasing an asset with an actual risk premium exceeding the one predicted by the CAPM and selling assets with CAPM risk premiums that exceed the actual one is a common decision rule for investors in financial markets