Financial Economics

4 Risky Decisions

LEC, SJTU

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Overview

Arrow-Debreu
general equilibrium,
welfare theorem,
representative agent

Radner economies
real/nominal assets,
market span,
risk-neutral prob.,
representative good

von Neumann Morgenstern
measures of risk
aversion, HARA class

Finance economy
SDF, CCAPM,
term structure

Data and the Puzzles

Empirical resolutions

Theoretical resolutions

Risky Decisions

- Probabilities and lotteries
- Expected Utility Theory
- Measures of risk preference
- Specialized class of utility functions

A very special ingredient: probabilities

- We defined commodities as being contingent on the state of the world- means that in principle we also cover decisions involving risk
- But risk has a special additional structure which other situations do not have: probabilities
- We have not explicitly made use of probabilities so far
  - The probabilities do affect preferences over contingent commodities, but so far we have not made this connection explicit
- Theory of decision-making under risk exploits this structure to get predictions about behavior of decision-makers

The St Petersburg Paradox

- Suppose someone offers you this gamble:
  - "I have a fair coin here. I'll flip it, and if it's tail I pay you $1 and the gamble is over. If it's head, I'll flip again. If it's tail then, I pay you $2, if not I'll flip again. With every round, I double the amount I will pay to you if it's tail."
- Sounds like a good deal. After all, you can't loose. So here's the question:
- How much are you willing to pay to take this gamble?

The expected value of the gamble

- The gamble is risky because the payoff is random. So, according to intuition, this risk should be taken into account, meaning, I will pay less than the expected payoff of the gamble
- So, if the expected payoff is X, I should be willing to pay at most X, possibly minus some risk premium
- BUT, the expected payoff of this gamble is INFINITE!

Infinite expected value

- With probability $1/2$ you get \$1
- With probability $1/4$ you get \$2
- With probability $1/8$ you get \$4
- .....
- The expected payoff is the sum of these payoffs, weighted with their probabilities, so

$$\sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t \cdot 2^{t-1} = \sum_{t=1}^{\infty} \frac{1}{2} = \infty$$

An infinitely valuable gamble?

- I should pay everything I own and more to purchase the right to take this gamble!
- Yet, in practice, no one is prepared to pay such a high price
- Why?
- Even though the expected payoff is infinite, the distribution of payoffs is not attractive: With 93% probability we get \$8 or less, with 99% probability we get \$64 or less

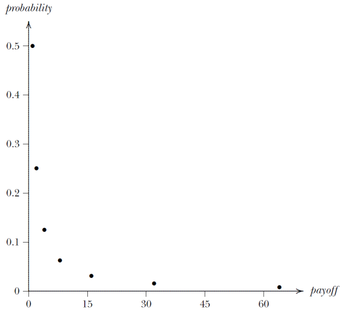


Figure 4.1. *Payoff distribution of the St. Petersburg gamble.*

What should we do?

- How can we decide in a rational fashion about such gambles (or investments)?
- Bernoulli suggests that large gains should be weighted less. He suggests to use the natural logarithm. [Cremer, another great mathematician of the time, suggests the square root.]

$$\sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^t \cdot \ln(2^{t-1}) = \ln(2) < \infty$$

- Bernoulli would have paid at most $e^{\ln(2)} = 2$ to participate in this gamble

Lotteries

- Suppose you are driving to work at Shanghai Jiao Tong University from Fudan
  - If you arrive on time prize (payoff)= x (prob.=95%)
  - If there is a traffic jam (prob=4.8%) you get nothing
  - If you have an accident (prob =0.2%) you get no payoff but also have to spend to repair your car.
  - This lottery can be written as: $[+x, 0.95; 0, 0.048; -y, 0.02]$
- Let us consider a finite set of outcomes: $[x_1, \dots, x_S]$
- The x_i's can be consumption bundles or in our case money - the x_i's themselves involve no uncertainty
- We define a lottery as:

$$[x_1, \pi_1; \dots; x_S, \pi_S], \quad \pi_s \geq 0, \sum_{s=1}^{S} \pi_s = 1$$

Preferences over Lotteries

- Let us call the set of all such lotteries as $\mathcal{L}$- we now assume that agents have preferences over this set
- So agents have a preference relation $\succ$ on $\mathcal{L}$ that satisfies the usual assumptions of ordinal utility theory
- Assumptions imply that we can represent such preferences by a continuous utility function $\mathcal{V} : \mathcal{L} \to \mathbb{R}$ so that

$$L \succ L' \iff \mathcal{V}(L) > \mathcal{V}(L')$$

- We also assume that people prefer more to less (in our case more money to less):

$$\pi_1 > 0, a > 0 \Rightarrow \mathcal{V}([x_1, \pi_1; x_2, \pi_2]) < \mathcal{V}([x_1 + a, \pi_1; x_2, \pi_2])$$

What is risk aversion?

- The expected value of a lottery is:

$$E[L] = \sum_{s=1}^{S} \pi_s x_s$$

- Consider the lottery $[E(L), 1]$- this lottery pays $E(L)$ with certainty. We call this degenerate lottery
- We define attitude to risk with reference to this lottery and how agents prefer outcomes relative to this lottery
  - *Risk Neutral*: $\mathcal{V}(L) = \mathcal{V}([E(L), 1])$ or the risk in the lottery L- variation in payoff between states is irrelevant to the agent- the agent cares only about the expectation of the prize
  - *Risk Averse*: $\mathcal{V}(L) < \mathcal{V}([E(L), 1])$ –here the agent would rather have the average prize $E(L)$ for sure than bear the risk in the lottery L
- A risk averse agent is willing to give up some wealth on average in order to avoid the randomness of the prize of L

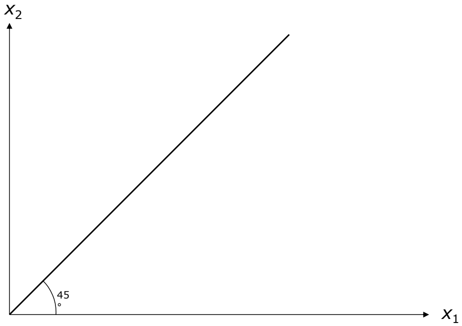
Certainty Equivalent

- Let $\mathcal{V}$ be some utility function on (set of all lotteries) and let L be some lottery with expected prize $E(L)$
- The **certainty equivalent** of L under $\mathcal{V}$ is defined as $\mathcal{V}([CE(L), 1]) = \mathcal{V}(L)$.
- $CE(L)$ is the level of non-random wealth that yields the same utility as the lottery L
- The **risk premium** is the difference between the expected prize of the lottery and its certainty equivalent: $RP(L) = E(L) - CE(L)$
- All of this is the same as ordinal utility theory and we have not used the additional structure in the probabilities- we now do this

The utility function $\mathcal{V}$

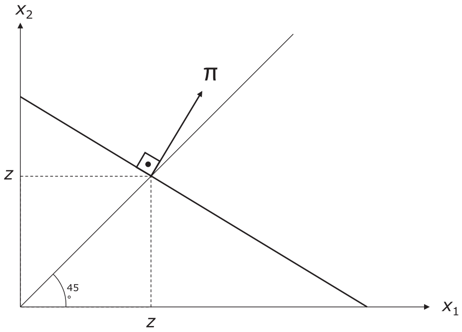
- In order to be able to draw indifference curves we will restrict attention to lotteries with only two possible outcomes, $[x_1, \pi_1; x_2, \pi_2]$
- Furthermore, we will also fix the probabilities (π_1, π_2), so that a lottery is fully described simply by the two payoffs (x_1, x_2). So a lottery is just a point in the plane
- From the ordinal utility function $\mathcal{V}$ we define a new function $\underline{\mathcal{V}}$ that takes only the payoffs as an argument, $\underline{\mathcal{V}}(x_1, x_2) = \mathcal{V}([x_1, \pi_1; x_2, \pi_2])$
- $\underline{\mathcal{V}}$ is very much like a utility function over two goods that we have used in Lecture 2. This makes it amenable to graphical analysis

Indifference curves



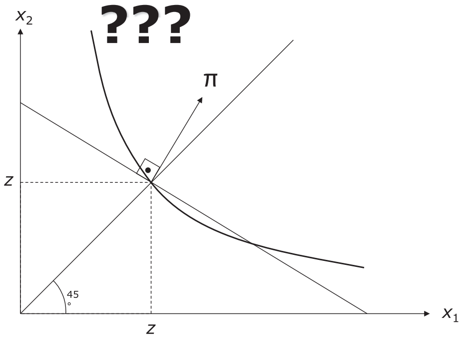
- Any point in this plane is a particular lottery
- Where is the set of risk-free lotteries?
- If $x_1 = x_2$, then the lottery contains no risk

Indifference curves



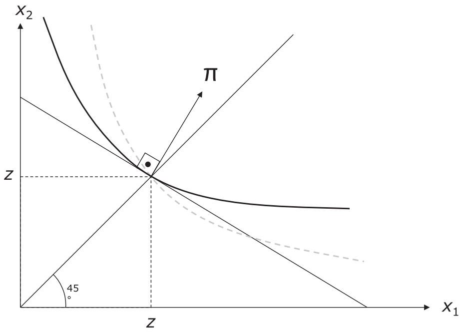
- Where is the set of lotteries with expected prize $E[L] = z$?
- It's a straight line, and the slope is given by the relative probabilities of the two states

Indifference curves



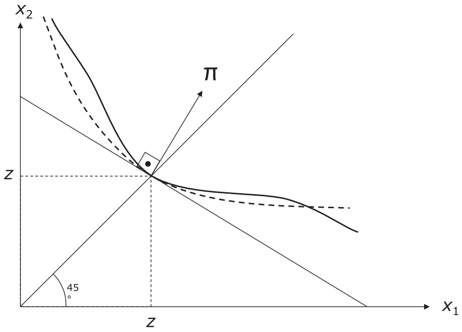
- Suppose the agent is risk averse. Where is the set of lotteries which are indifferent to (z, z)?
- That's not right! Note that there are risky lotteries with smaller expected prize and which are preferred

Indifference curves



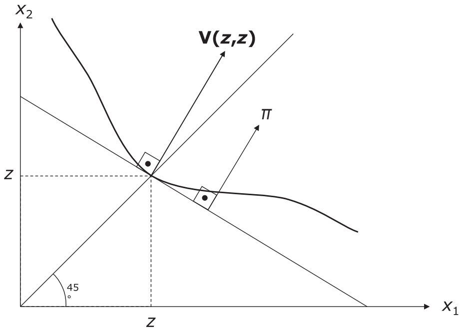
- So the indifference curve must be tangent to the iso-expected-prize line
- This is a direct implication of risk-aversion alone

Indifference curves



- But risk-aversion does not imply convexity
- This indifference curve is also compatibe with risk-aversion

Indifference curves



- The tangency implies that the gradient of $\underline{\mathcal{V}}$ at the point (z, z) is collinear to π
- Formally, $\nabla \underline{\mathcal{V}}(z, z) = \lambda \pi$, for some $\lambda > 0$

Indifference curves

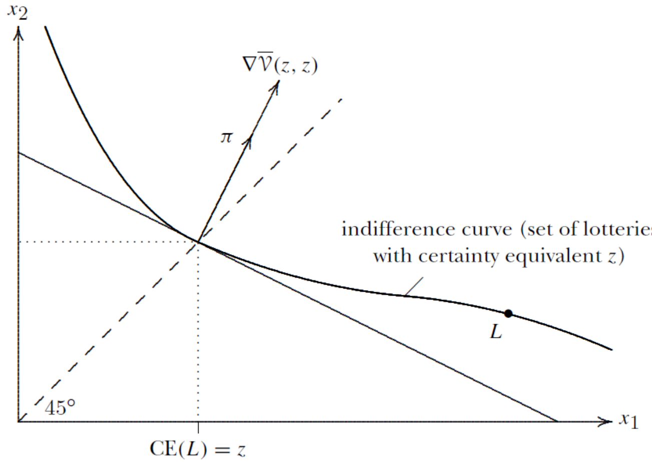


Figure 4.3. *An indifference curve and the certainty equivalent.*

Indifference curves

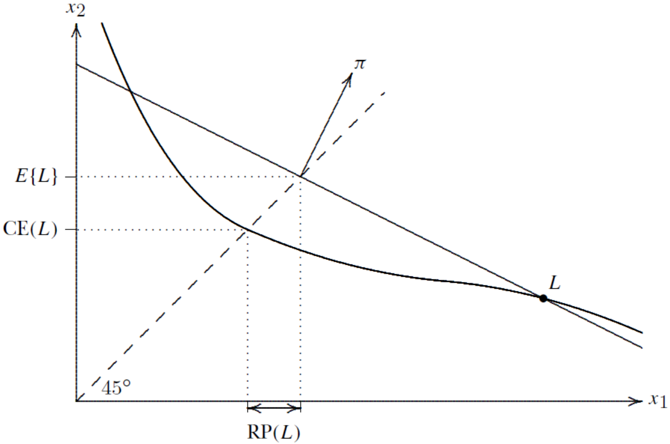


Figure 4.4. *Certainty equivalent and risk premium.*

What we are after: an *expected utility representation*

- So far we have used ordinal utility theory and we now add the idea of probabilities
- We want to represent agent's preferences by evaluating the expected utility of a lottery
- We need a function v that maps the single outcome x_s to some real number $v(x_s)$ and then we compute the expected value of v.
- Formally function v is the expected utility representation of $\mathcal{V}$ if:

$$\mathcal{V}([x_1, \pi_1; \ldots; x_S, \pi_S]) = \sum_{s=1}^{S} \pi_s v(x_s)$$

- von Neumann and Morgenstern first developed the use of an expected utility under some conditions- lets look at these briefly

vNM Axioms: State Independence

- von Neumann and Morgenstern's have presented a model that allows the use of an expected utility under some conditions
- The first assumption is state independence
- All that matters to an agent is the statistical distribution of outcomes.
- A state is just a label and has no particular meaning and are interchangeable (as in x and y in the diagram)

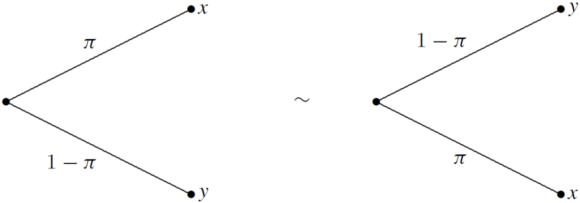


Figure 4.5. *State-independence.*

vNM Axioms: Consequentialism

- Consider a lottery L whose prizes are further lotteries L_1 and L_2: $L = [L_1, \pi_1; L_2, \pi_2]$- a compound lottery
- We assume that an agent is indifferent between L and a one-shot lottery with four possible prizes and compounded probabilities
- An agent is indifferent between the two lotteries shown in the diagram below
- Agents are only interested in the distribution of the resulting prize but not in the process of gambling itself

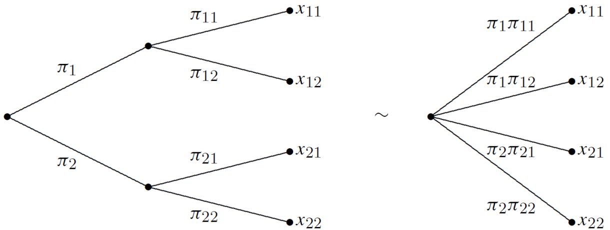


Figure 4.6. *Consequentialism.*

vNM Axioms: Irrelevance of Common Alternatives

- This axiom says that the ranking of two lotteries should depend only on those outcomes where they differ
- If L_2 is better than L_1 and we compound each of these lotteries with some third common outcome x then it should be true that $[L_2, \pi; x, 1-\pi]$ is still better than $[L_1, \pi; x, 1-\pi]$. The common alternative x should not matter

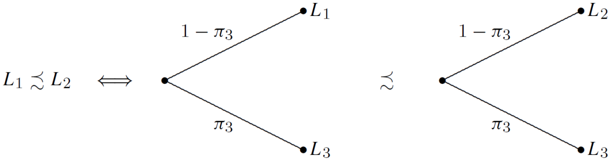


Figure 4.7. *Irrelevance of common alternatives.*

vNM Utility Theory - Some Discussion

- State-independence, consequentialism and the irrelevance of common alternatives + the assumptions on preferences give rise to the famous results of vNM
- The utility function $\mathcal{V}$ has an expected utility representation v such that:

$$\mathcal{V}([x_1, \pi_1; \ldots; x_S, \pi_S]) = \sum_{s=1}^{S} \pi_s v(x_s)$$

- The utility function is on the space of lotteries $\mathcal{L}$ which represents the preference relation between lotteries and is an ordinal utility function
  - $\mathcal{V}(L)$ is an ordinal measure of satisfaction and can be compared only in the sense of ranking lotteries
  - $\mathcal{V}$ is also invariant to monotonic transformations

vNM Utility Theory - Some More Discussion

- The vNM utility function v has more structure
  - It represents $\mathcal{V}$ as a linear function of probabilities
  - As a result, v is not invariant under an arbitrary monotonic transformation
  - It is invariant only under positive affine transformations: $f(x) = a + bx, a > 0, b > 0$
- Hence vNM utility is cardinal
- Cardinal numbers are measurements that are ordinal but whose difference can also be ordered
  - With cardinal utility we can have: $v(x_1) - v(x_2) > v(x_3) - v(x_4)$, meaning that x_1 is better than x_2 "by a larger amount" than x_3 is better than x_4

Risk-aversion and Concavity-I

- The certainty equivalent is the level of wealth that gives the same utility as the lottery on average. Formally:

$$v(CE(x)) = E[v(x)]$$

- We can explicitly solve for the CE as: $CE(x) = v^{-1}(E[v(x)])$

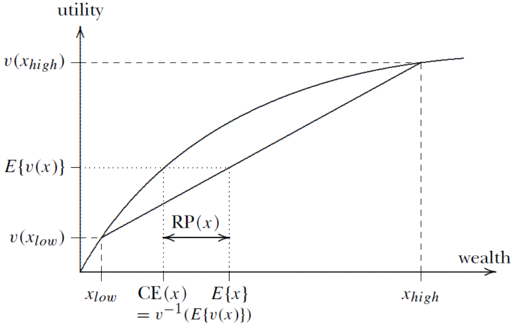


Figure 4.8. *A risk-averse NM utility is concave.*

Risk-aversion and Concavity-II

- An agent is risk averse if v is a concave function
- Jensen' s inequality: strict convex combination of two values of a function is strictly below the graph of the function then the function is concave
- The risk premium is therefore positive and the agent is risk averse if v is strictly concave
- If $v'' = 0$, then $CE(x) = E[x]$ and the $RP = 0$ or risk neutrality

An insurance problem

- Consider an insurance problem:
  - d amount of damage
  - π probability of damage
  - μ insurance premium for full coverage
  - c amount of coverage

$$\max_c (1-\pi)v(w-c\mu) + \pi v(w-c\mu-(1-c)d)$$

- The FOC of this problem is

$$\frac{1-\pi}{\pi} \cdot \frac{v'(w-c\mu)}{v'(w-c\mu-(1-c)d)} = \frac{d-\mu}{\mu}$$

An insurance problem

- Full coverage ($c = 1$) implies

$$\frac{1-\pi}{\pi} = \frac{d-\mu}{\mu} \Rightarrow \mu = \pi d$$

- Full coverage is optimal only if the premium is statistically fair
- Suppose the premium is not fair. Let $\mu = (1+m)\pi d$, and $m > 0$ be the insurance company's markup. Then, $\frac{1-\pi}{\pi} > \frac{d-\mu}{\mu}$. By FOC

$$v'(w-c\mu) < v'(w-c\mu-(1-c)d) \Rightarrow w-c\mu > w-c\mu-(1-c)d \Rightarrow c < 1$$

An insurance problem

- If the insurance premium is not fair, it is optimal not to fully insure
- In fact, if the premium is large enough (m_0), no coverage is optimal
- The FOC, with μ substituted by $(1+m)\pi d$, is

$$\frac{1-\pi}{\pi} \cdot \frac{v'(w - c(1+m)\pi d)}{v'(w - c(1+m)\pi d - (1-c)d)} = \frac{d - (1+m)\pi d}{(1+m)\pi d}$$

- We extract m_0 by setting $c = 0$

$$\frac{1-\pi}{\pi} \cdot \frac{v'(w)}{v'(w-d)} = \frac{d - (1+m_0)\pi d}{(1+m_0)\pi d}$$

$$\Rightarrow m_0 = \frac{(1-\pi)(v'(w-d) - v'(w))}{(1-\pi)v'(w) + \pi v'(w-d)}$$

An insurance problem

$$m_0 = \frac{(1-\pi)(v'(w-d) - v'(w))}{(1-\pi)v'(w) + \pi v'(w-d)}$$

- If $\mu = (1+m_0)\pi d$, the agent is just indiff between insuring and carrying the whole risk, when the risk (d) approaching zero
- Thus, $w - (1+m_0)\pi d$ is the certainty equivalent
- It is clear that m_0 vanishes as the risk becomes smaller, $\pi d \to 0$
- But the relative speed of convergence is not so clear: how fast does m_0 vanish compared to πd?

$$\lim_{d \to 0} \frac{m_0}{\pi d} = ?$$

Absolute Risk Aversion

$$\lim_{d \to 0} \frac{m_0}{\pi d} = -\frac{1-\pi}{\pi} \cdot \lim_{d \to 0} \frac{(v'(w) - v'(w-d))/d}{(1-\pi)v'(w) + \pi v'(w-d)}$$

- For symmetric risks ($\pi = 1/2$) we thus get

$$\lim_{d \to 0} \frac{m_0}{\pi d} = -\frac{v''(w)}{v'(w)} :- A(w)$$

- This is the celebrated coefficient of absolute risk aversion, discovered by Pratt and by Arrow
- We see here that it is a measure for the size of the risk premium for an infinitesimal risk

Absolute Risk Aversion

- We define the coefficient of Absolute Risk Aversion (ARA) as a local measure of the degree that an agent dislikes risk

$$A(w) :- -\frac{v''(w)}{v'(w)}$$

- A has many useful properties:
  - It is invariant under an affine transformation. This means we can use the ARA then for interpersonal comparisons
  - Suppose vNM utility function v is more concave than u , then ARA for $v(w)$ is larger than the ARA for $u(w)$

CARA-DARA-IARA

- A utility function v exhibits constant absolute risk aversion or CARA if ARA does not depend on wealth or $A'(w) = 0$.
- v exhibits decreasing absolute risk aversion or DARA if richer people are less absolutely risk averse than poorer ones or $A'(w) < 0$.
- v exhibits increasing absolute risk aversion or IARA if $A'(w) > 0$.
- What do these mean in economic terms?
  - Consider a simple binary lottery - you cannot win anything but can lose \$10 with 50% probability
  - CARA $\Rightarrow$ millionaire requires the same payment to enter this lottery as a beggar would
  - IARA $\Rightarrow$ millionaire requires a larger payment than the beggar!
  - DARA $\Rightarrow$ millionaire takes it for a smaller payment than a beggar - most realistic case

Relative Risk Aversion

- Consider another simple binary lottery - instead of losing \$10 with 50% probability now we have a 50% probability of losing your wealth
  - For the beggar this amount to losing 50 cents, for the millionaire it may be in \$100,000
  - Who requires a larger amount up front, in terms of percentage of his wealth, to enter this gamble? Not easy to answer?
  - Suppose the millionaire requires \$70,000 - this is not unrealistic and the beggar requires 30 cents - also probable - then the millionaire requires a larger percentage of his wealth than the beggar $\Rightarrow$ millionaire is thus more relatively risk averse than the beggar.
- This is measured as Coefficient of RRA: $R(w) = w \cdot A(w)$
- If R is independent of wealth then we call that CRRA utility functions

Prudence

- Coefficients of risk aversion measure the disutility arising from a small amount of risk imposed on agents or how much an agent dislikes risk
- Coefficients do not tell us about how the behavior of agents changes when we vary the amount of risk the agent is forced to bear
  - Example: It may be reasonable for agents to accumulate some "precautionary" saving when facing more uncertainty
  - More risk induces a more prudent agent to accumulate precautionary savings
- Kimball's coefficient of absolute prudence:

$$P(w) = -\frac{v'''}{v''}$$

- An agent is prudent if this coefficient is positive
- The precautionary motive is important because it means that agents save more when faced with more uncertainty
- Prudence seems uncontroversial, because it is weaker than DARA

Empirical Estimates

- Many studies have tried to obtain estimates of these coefficients using real-world data
- Friend and Blume (1975): study U.S. household survey data in an attempt to recover the underlying preferences. Evidence for DARA and almost CRRA, with $R \approx 2$
- Tenorio and Battalio (2003): TV game show in which large amounts of money are at stake. Estimate relative risk aversion between 0.6 and 1.5
- Abdulkadri and Langenmeier (2000): farm household consumption data. They find significantly more risk aversion
- Van Praag and Booji (2003): survey-based study done by a Dutch newspaper. They find that relative risk aversion is close to log-normally distributed, with a mean of 3.78

Introspection

- In order to get a feeling for what different levels of risk aversion actually mean, it may be helpful to find out what your own personal coefficient of risk aversion is
- You can do that by working through Box 4.6 of the book, or by using the electronic equivalent available from the website

Frequently Used Utility Functions

- Utility functions that (i) strictly increasing (ii) strictly concave (iii) DARA or $A'(w) < 0$ (iv) not too large relative risk aversion $0 < R(w) < 4$ for all w are the properties that are most plausible

name	formula	A	R	P	a	b
affine	$\gamma_0 + \gamma_1 y$	0	0	undef	undef	undef
quadratic	$\gamma_0 y - \gamma_1 y^2$	incr	incr	0	$\gamma_0/(2\gamma_1)$	-1
exponential	$-\frac{1}{\gamma} e^{-\gamma y}$	γ	incr	γ	$1/\gamma$	0
power	$\frac{1}{1-\gamma} y^{1-\gamma}$	decr	γ	decr	0	$1/\gamma$
Bernoulli	$\ln y$	decr	1	decr	0	1

- A, R, and P denote absolute risk aversion, relative risk aversion, and prudence. a and b will be explained later
- All these belong to the class of HARA functions

The HARA Class

- Most of the plausible utility functions belong to the HARA or hyperbolic absolute risk aversion (or linear risk tolerance utility function) class
- Define absolute risk tolerance as the reciprocal of absolute risk aversion, $T :- 1/A$
- u is HARA if T is an affine function, $T(y) = a + by$
- Merton shows that a utility function v is HARA if and only if it is an affine transformation of:

$$v(y) :- \begin{cases} \ln(y+a), & \text{if } b = 1, \\ -ae^{-y/a}, & \text{if } b = 0, \\ (b-1)^{-1}(a+by)^{(b-1)/b}, & \text{otherwise.} \end{cases}$$

- DARA $\Rightarrow b > 0$; CARA $\Rightarrow b = 0$; IARA $\Rightarrow b < 0$, v is CRRA if $a = 0$.
- Most results in finance rely on assumption of HARA utility - whether these are realistic is another matter.

Risky Decisions

- Probabilities and lotteries
- Expected Utility Theory
- Measures of risk preference
- Specialized class of utility functions