Financial Economics
Problem Set 2

LEC, SJTU
Winter 2024

1. Consider an investor with mean-variance preferences represented by the utility function $V(\mu, \sigma^2)$ which is increasing in the mean μ and decreasing in the variance σ^2. The investor can choose a portfolio of K assets $a = (a_1, \ldots, a_K)$ yielding retllrns $r_k = (r_1^k, \ldots, r_S^k)$, $k = 1, \ldots, K$, in S states of the world. The price of asset k is q_k, $k = 1, \ldots, K$. The probability distribution over states is given by the vector $(p_1, \ldots, p_S)$.

(a) Derive the mean $\mu(a)$ and the variance $\sigma^2(a)$ of a portfolio a.

(b) Derive the first-order conditions for an optimal portfolio.

(c) Show that the optimal portfolio minimizes the variance over all portfolios with the same mean value that satisfy the budget constraint.

2. Consider an economy with trade in K assets at market prices $(q_1, \ldots, q_K)$. Assets pay returns $r_s = (r_s^1, \ldots, r_s^K)$ in each of S states that occur with probabilities $(p_1, \ldots, p_S)$.

(a) Write down the optimization problem for the choice of a portfolio that minimizes the variance subject to the constraint that it satisfies the budget constraint and that it achieves a certain level of mean return.

(b) Derive the first-order conditions for the variance-minimizing portfolio. Are these conditions sufficient as well?

(c) Assume that asset K is riskless with return R and price $q_K \equiv 1$. Show that the first-order conditions for the $K-1$ risky assets can be written in the following form:

$$[\mu_k - q_k \cdot R] = \frac{\sigma(k, a)}{\sigma^2(a)} \cdot [\mu(a) - R \cdot \sum_{k=1}^{K} q_k \cdot a_k]$$

3. Let ρ be the risk-free interest rate and $E\{R\}$ be the expected yield of some (possibly risky) asset. Let the asset we are looking at be an Arrow security that pays out in a state when aggregate endowment is particularly low. This security is clearly risky. Is $E\{R\} > \rho$, $E\{R\} = \rho$, or $E\{R\} < \rho$, and why?

4. Consider a finance economy with two states with probabilities π and $1 - \pi$, respectively, and utility function of income $\ln(x)$. Current consumption does not enter agents' utilities; they are interested only in consumption tomorrow.

Compute equilibrium prices, equilibrium allocations, and gains from trade (in terms of ex ante expected utility of the agents), assuming, in turn, that

$$w(1) := \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad w(2) := \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \text{with } \pi = {}^1\!/_2, \qquad (a)$$

$$w(1) := \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad w(2) := \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \text{with } \pi = {}^2\!/_3, \qquad (b)$$

$$w(1) := \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad w(2) := \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \qquad \text{with } \pi = {}^1\!/_2. \qquad (c)$$

Case (a) is completely symmetric. There is scope for full mutual insurance. Does it happen? Case (b) is similar to case (a) in the sense that there is still no aggregate uncertainty. However, there is an asymmetry between the two agents because agent 2 is richer than agent 1 on average. Will there be full insurance? Case (c) is interesting: now there is aggregate uncertainty, but it is all borne by agent 1. Agent 2 faces no endowment uncertainty. Will he offer (partial) insurance to agent 1?

5. Consider case (b) of the previous problem, but assume that agent 1's assessment of the probability of state 1 is $\pi(1) :- 2/3$, and agent 2's assessment is $\pi(2) :- 1/3$. Compute the equilibrium.

6. Derive the CCAPM pricing formula and risk-free interest rate when the representative agent has Bernoulli utility function ($u(y) = \ln y$).